arXiv:0709.3218v2 [math.AG] 17 Jan 2009

Positivity of Chern classes for Reflexive Sheaves on $\mathbf{P}^n$

Cristina Bertone-Margherita Roggero
Dipartimento di Matematica dell'Università
Via Carlo Alberto 10
10123 Torino, Italy
margherita.roggero@unito.it
cristinabertone@hotmail.it

Abstract

It is well known that the Chern classes c_i of a rank n vector bundle on $\mathbf{P}^N$, generated by global sections, are non-negative if $i \leq n$ and vanish otherwise. This paper deals with the following question: does the above result hold for the wider class of reflexive sheaves? We show that the Chern numbers c_i with $i \geq 4$ can be arbitrarily negative for reflexive sheaves of any rank; on the contrary for $i \leq 3$ we show positivity of the c_i with weaker hypothesis. We obtain lower bounds for c_1, c_2 and c_3 for every reflexive sheaf $\mathcal{F}$ which is generated by $H^0\mathcal{F}$ on some non-empty open subset and completely classify sheaves for which either of them reach the minimum allowed, or some value close to it.

1 Introduction

In this paper we investigate some general conditions that ensure the positivity of Chern classes for reflexive sheaves of any rank on the projective space $\mathbf{P}^N$. There are some classical results about vector bundles:

If a rank n vector bundle $\mathcal{F}$ on $\mathbf{P}^N$ is generated by global sections, then its Chern classes c_i are non-negative if $i \leq n$, while the following ones vanish (see [1], Example 12.1.7).

We would like to weaken both hypotheses, considering the wider class of reflexive sheaves (instead of vector bundles) generated by global sections on some (non-empty) open subset.

In this new context, the situation is immediately more complicated. First of all, a rank n reflexive sheaf has in general non-zero Chern classes c_i also for $i > n$. Moreover, it is not difficult to obtain for every pair (n, i) (both ≥ 4) examples of rank n reflexive sheaves on $\mathbf{P}^N$, generated by global sections, having negative c_i (see Example 5.7).

So, we can not expect to control the positivity of the i-th Chern class for every reflexive sheaf when $i \geq 4$, even if i is lower than the rank.

The different behavior of a general reflexive sheaf $\mathcal{F}$ with respect to a vector bundle is clearly due to the presence of its "singular locus" $\mathcal{S}$, that is the set of points where $\mathcal{F}$ is not locally free; $\mathcal{S}$ is a closed subset of codimension ≥ 3, so that if $i > 3$ the i-th Chern class $c_i(\mathcal{F})$, which is given by a cycle of codimension i, can have components contained in $\mathcal{S}$. So we cannot expect that $c_i(\mathcal{F})$ is necessarily positive, even if the locally free sheaf $\mathcal{F}_U$, the restriction of $\mathcal{F}$ to $U = \mathbf{P}^n \setminus \mathcal{S}$, is generated by global sections.

We might think to apply the same argument also to the third Chern class c_3 and even to the lower ones c_1 and c_2 in case the reflexive sheaf (or even the bundle) is not globally generated on some closed subset of small codimension. Thus, it is a little surprising to discover that, on the contrary, c_1, c_2 and c_3 are positive under the above weaker conditions.

[0]Mathematics Subject Classification 2000: 14F05, 14F17, 14Jxx
Keywords: rank 2 bundles, subcanonical surfaces.

In fact, in §3 we obtain the above quoted positivity results for the first and second Chern classes of a rank n reflexive sheaf $\mathcal{F}$, generated by global sections on a non-empty open subset of $\mathbf{P}^N$, as a consequence of more general inequalities involving c_1 and c_2. More precisely,:

Theorem A *If $\mathcal{F}$ is not a direct sum of line bundles and it has a proper subsheaf isomorphic to $\oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^N}(\alpha_i)$ where $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, then:*

$$c_1 \geq \sum \alpha_i + 1 \quad \text{and} \quad c_2(\mathcal{F}) \geq \sum_{i<j} \alpha_i\alpha_j + \sum_{i\neq 2} \alpha_i + 1.$$

Moreover

$$c_1 \geq \sum \alpha_i + 2 \quad \text{and} \quad c_2(\mathcal{F}) \geq \sum_{i<j} \alpha_i\alpha_j + 2\sum_{i\neq 2} \alpha_i + 2$$

unless $\mathcal{F}$ has a short free resolution of the type:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(\beta - 1) \longrightarrow \oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \oplus \mathcal{O}_{\mathbf{P}^N}(\beta) \longrightarrow \mathcal{F} \longrightarrow 0$$

(see Corollary 3.6 and Corollary 3.8).

In §4 we obtain similar results on c_1 and c_2 using a slightly different set of hypotheses, also involving the general splitting type of $\mathcal{F}$.

Finally in §5 we obtain similar results about the third Chern class c_3 of a rank n reflexive sheaf on $\mathbf{P}^N$:

Theorem B *if $\mathcal{F}$ is generated by global sections outside a closed subset of codimension ≥ 3, then $c_3(\mathcal{F}) \geq 0$ and equality $c_3(\mathcal{F}) = 0$ can hold only if either $N = 3$ and $\mathcal{F}$ is a vector bundle or $N \geq 4$ and $\mathcal{F}_H$ is a vector bundle for every general linear subspace $H \cong \mathbf{P}^3$ in $\mathbf{P}^N$*

(see Theorem 5.2). Under some additional condition on the homological dimension of $\mathcal{F}$, c_3 can vanish only if $\mathcal{F}$ is a bundle:

Theorem C *if $\mathrm{hd}(\mathcal{F}) \leq 1$ and $\mathrm{hd}(\mathcal{F}^\vee) \leq 1$, then $c_3(\mathcal{F}) = 0$ only if $\mathcal{F}$ is a vector bundle having a direct summand $\mathcal{O}_{\mathbf{P}^N}^r$, for some $r \geq n - 2 - h^1\mathcal{F}(-c_1)$*

(see Corollary 5.3). This extends to sheaves of any rank on projective spaces of any dimension a well known property for rank 2 reflexive sheaves on $\mathbf{P}^3$ (see [3], Proposition 2.6).

2 Notation and preliminary results

In this paragraph we introduce some general facts on reflexive sheaves and Chern classes that we will use in the paper and especially we study the singular locus of a reflexive sheaf $\mathcal{F}$ and its maximal free subsheaves (see Definition 2.7) that will be the main tool in the proofs.

In what follows, we consider an algebraically closed field k of characteristic 0. Actually, the results of §3 hold more generally over a ground field of any characteristic, while we need the characteristic 0 in §4 and §5 in order to use Grothendieck's Theorem for vector bundles on a line and Generic Smoothness.

$\mathbf{P}^N$ is the projective space of dimension N over k. As usual, if $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^N$, we will denote by $h^i(\mathcal{F})$ the dimension of the i-th cohomology module $H^i(\mathcal{F})$ as a k-vector space and by $H^i_*\mathcal{F}$ the direct sum $\oplus_{n\in\mathbb{Z}} H^i\mathcal{F}(n)$; in particular $H^0_*\mathcal{O}_{\mathbf{P}^N} = k[X_0, \ldots, X_n]$ and for $\mathcal{F}$ coherent sheaf, $H^0_*\mathcal{F}$ has a natural structure of $H^0_*\mathcal{O}_{\mathbf{P}^N}$-module; if Y is a subvariety (that is a closed subscheme) in $\mathbf{P}^N$, we will denote by $\deg_2(Y)$ the degree of the codimension 2 (may be reducible or not reduced) component of Y.

We recall some basic properties of Chern classes and reflexive sheaves.

1. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^N$, we denote by $c_i(\mathcal{F})$ or simply c_i $(i = 1, \ldots N)$ its Chern classes that we think as integers and by

$$C_t(\mathcal{F}) = 1 + c_1(\mathcal{F})t + \cdots + c_{N-1}(\mathcal{F})t^{N-1} + c_N(\mathcal{F})t^N$$

its Chern polynomial. If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence, then $C_t(\mathcal{F}) = C_t(\mathcal{F}')C_t(\mathcal{F}'')$ in $\mathbb{Z}[t]/(t^{N+1})$.

If c_i are the Chern classes of a rank r coherent sheaf $\mathcal{F}$, the Chern classes of $\mathcal{F}(l)$ are given by:

$$c_i(\mathcal{F}(l)) = c_i + (r-i+1)lc_{i-1} + \binom{r-i+2}{2} l^2 c_{i-2} + \cdots + \binom{r}{i} l^i. \tag{1}$$

2. We say that a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^N$ is *reflexive* if the canonical morphism $\mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism, where $\mathcal{F}^\vee$ is the dual sheaf, that is $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})$. We refer to [3] and to [9] for general facts about reflexives sheaves, especially about rank 2 reflexive sheaves on $\mathbf{P}^3$ and $\mathbf{P}^4$. We only recall some of them that we will use more often. The dual of every sheaf is reflexive and, for every integer l, $\mathcal{O}_{\mathbf{P}^N}(l)$ is the the only rank 1 reflexive sheaf on $\mathbf{P}^N$ with $c_1 = l$.

   A reflexive sheaf $\mathcal{F}$ is locally free except at most on a closed subset $\mathcal{S}(\mathcal{F})$ of codimension ≥ 3, its **singular locus**. Then, reflexive sheaves on $\mathbf{P}^1$ and $\mathbf{P}^2$ are in fact vector bundles, while on $\mathbf{P}^N$ if $N \geq 3$ there are reflexive sheaves which are not vector bundles: for every irreducible, codimension 2, subvariety Y in $\mathbf{P}^N$ and a general section of $\omega_Y(t)$ ($t \gg 0$) we can construct a non-locally free reflexive sheaf of rank 2 as an extension:

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{F} \to \mathcal{I}_Y(b) \to 0$$

   where b depends on Y and t (see [3], Theorem 4.1).

   In the following we will use several times the following general facts:

Lemma 2.1 *Let $\mathcal{F}$ be a torsion free (respectively: reflexive, locally free) sheaf on $\mathbf{P}^N$. If H is a general linear subspace of dimension r in $\mathbf{P}^N$, ($1 \leq r \leq N-1$), then :*

(i) *H is "regular" with respect to $\mathcal{F}$ that is* $\mathrm{Tor}^1(\mathcal{F}, \mathcal{O}_H) = 0$

(ii) *$\mathcal{F}_H$ is a torsion free (reflexive, locally free) sheaf on H;*

(iii) *the dual of $\mathcal{F}_H$ as a sheaf on H is isomorphic to $(\mathcal{F}^\vee)_H$;*

(iv) *if $\mathcal{F}$ is reflexive, the singular locus $\mathcal{S}(\mathcal{F}_H)$ as a sheaf on H is precisely $\mathcal{S}(\mathcal{F}) \cap H$.*

(v) *for every $i \leq r$, $c_i(\mathcal{F}_H) = c_i(\mathcal{F})$ (where $\mathcal{F}_H$ is considered as a sheaf on H).*

Proof: It is sufficient to prove the results for a general hyperplane H and use induction on N.

(i) and *(ii)* are special cases of more general statements proved in [5] (see Lemma 1.1.12 and Corollary 1.1.14 *iii)*). In the present context $\mathcal{F}$ has "codimension" $c = 0$, so that "$\mathcal{F}$ pure (reflexive) of codimension 0" is equivalent to $\mathcal{F}$ torsion free (reflexive), and "$\mathcal{F}_H$ pure (reflexive) of codimension 1 as a sheaf on $\mathbf{P}^N$ " is equivalent to $\mathcal{F}_H$ torsion free (reflexive) as a sheaf on H.

For *(iii)*, that is the isomorphism $(\mathcal{F}^\vee)_H \cong (\mathcal{F}_H)^\vee$, see again [5], the remark after Definition 1.1.7.

(iv) As in *ii)* we can see that $\mathcal{S}(\mathcal{F}_H) \subseteq \mathcal{S}(\mathcal{F}) \cap H$, because, for a general H, $\mathcal{F}_H$ is locally free (as a sheaf on H) where $\mathcal{F}$ is (as a sheaf on $\mathbf{P}^N$). Moreover $\mathbf{P}^N$ is a regular variety and H is regular with respect to $\mathcal{F}$; then for every point x contained in the hyperplane H, $\dim(\mathcal{O}_{\mathbf{P}^N,x}) - \mathrm{depth}(\mathcal{F}_x)$ as $\mathcal{O}_{\mathbf{P}^N,x}$-modules, coincides with $\dim(\mathcal{O}_{H,x}) - \mathrm{depth}(\mathcal{F}_{H,x})$ as $\mathcal{O}_{H,x}$-modules, so that x is a singular point for $\mathcal{F}$ if and only if it is for $\mathcal{F}_H$.

(v) Fix a free resolution of $\mathcal{F}$; for a general H it restricts to a free resolution of $\mathcal{F}_H$ on H. Now it is sufficient to observe that the equality holds for all free sheaves and use the multiplicativity of Chern polynomials.

$\diamond$

Lemma 2.2 *Assume that the ground field k has characteristic 0. Let $\mathcal{F}$ be a rank r reflexive sheaf on $\mathbf{P}^N$ generated by global sections outside a closed subset Z of codimension ≥ 2. Then $n-1$ general global sections degenerate on a closed subset Y of codimension ≥ 2, generically smooth outside Z and $\mathcal{S}$.*

Proof: On the open subset $U = \mathbf{P}^N - (Z \cup \mathcal{S})$ of $\mathbf{P}^N$ the restriction map $H^0\mathcal{F} \to H^0\mathcal{F}_U$ is in fact a bijection ([3], Proposition 1.6) and so the bundle $\mathcal{F}_U$ on U is generated by its global sections too (so that $h^0\mathcal{F} = h^0\mathcal{F}_U \geq r$).

Take $n = r-1$ general global sections $s_1, \dots, s_{r-1}$ and consider their degeneracy locus $Y' = \{x \in U \text{ s.t. } \dim Span(s_1(x), \dots, s_{r-1}(x)) \leq r-2\}$ on U. If both $m = r-2$ and $m = r-3$ satisfy the inequality $max\{0, 2r-1-h^0\mathcal{F}_U\} \leq m \leq r-1$, we can apply Remark 6 of [6] to the bundle $\mathcal{F}_U$ and to the vector space $V = H^0\mathcal{F}_U$ and conclude that either Y' is empty or it has pure codimension 2 and it is smooth outside the subset $Y'' = \{x \in U \text{ s.t. } \dim Span(s_1(x), \dots, s_{r-1}(x)) \leq r-3\}$ of codimension ≥ 3.

If $h^0\mathcal{F}_U = r$, the degeneracy locus on U of $r-1$ general sections is empty. If $h^0\mathcal{F}_U = r+1$, we can deduce from [6] that either Y' is empty or $\mathrm{codim}(Y') = 2$ and Y' is smooth outside Y''. In order to prove that $\mathrm{codim}(Y'') \geq 3$ we can apply [1], Example 14.3.2 (d) to $\mathcal{F}_U$ with $p = h = 3$ and $\lambda' = (3, 0 \dots, 0)$ (observe that the first $r-1$ of $2r$ general sections are general too).

Finally, if Y is the degeneracy locus on $\mathbf{P}^N$ of $s_1, \dots, s_{r-1}$, then $Y \cap U = Y'$, so that $\mathrm{codim}(Y) \geq 2$ and it is smooth outside Z, $\mathcal{S}$ and Y''.

$\diamond$

3. Assume that the ground field k has characteristic 0. For any rank n reflexive sheaf $\mathcal{F}$, we denote by $st(\mathcal{F})$ the **splitting type** of $\mathcal{F}$ that is the sequence of integers $(b_1, \dots, b_n)$, (we will assume $b_1 \geq \dots \geq b_n$) such that for every general line L in $\mathbf{P}^N$, $\mathcal{F}_L = \oplus_{i=1}^n \mathcal{O}_L(b_i)$; recall that $c_1(\mathcal{F}) = b_1 + \dots + b_n$.

Lemma 2.3 *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^N$ with Chern classes c_i and let Y be a subvariety of $\mathbf{P}^N$ of codimension ≥ 2. If there is an exact sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(q) \longrightarrow 0 \tag{2}$$

then $\mathcal{G}$ is reflexive and :

$$c_1(\mathcal{F}) = c_1(\mathcal{G}) + q \qquad c_2(\mathcal{F}) = c_2(\mathcal{G}) + \deg_2(Y) + c_1(\mathcal{G})q. \tag{3}$$

Furthermore, if $\mathcal{G}$ is a vector bundle, then Y is empty or it has pure codimension 2.

Proof: The equalities on c_1 and c_2 can be easily computed by (2) using 1. in §2: in fact $c_1(\mathcal{F}) = c_1(\mathcal{G}) + c_1(\mathcal{I}_Y(q))$ and $c_2(\mathcal{F}) = c_2(\mathcal{G}) + c_2(\mathcal{I}_Y(q)) + c_1(\mathcal{G})c_1(\mathcal{I}_Y(q))$ where $c_1(\mathcal{I}_Y(q)) = q$ and $c_2(\mathcal{I}_Y(q)) = \deg_2(Y)$.

In order to show that $\mathcal{G}$ is reflexive it is sufficient to observe that $\mathcal{I}_Y$ is torsion-free and use [3] Corollary 1.5.

Finally, suppose that $\mathcal{G}$ is a vector bundle and let p be a codimension ≥ 3 point in $\mathbf{P}^N$. Then $\mathrm{depth}(\mathcal{G}_p) = \mathrm{codim}(p) \geq 3$ and $\mathrm{depth}(\mathcal{F}_p) \geq 2$, because $\mathcal{G}$ is locally free and $\mathcal{F}$ is reflexive (see [3] Proposition 1.3). Localizing the exact sequence (2) at the point p and using the characterization of depth in terms of non-vanishing of Ext^i ([8] Theorem 28), we see that $\mathrm{depth}(\mathcal{I}_{Yp}) = \mathrm{depth}(\mathcal{F}_p) \geq 2$, so that $\mathrm{depth}(\mathcal{O}_{Yp}) \geq 1$ so that p is not an associated prime to Y.

$\diamond$

Definition 2.4 *Let $\mathcal{H}$ be a coherent sheaf on $\mathbf{P}^N$.*
We say that $\mathcal{H}$ has m **independent global sections** *if there is an injective map:*

$$\phi : \mathcal{O}_{\mathbf{P}^N}^m \to \mathcal{H}.$$

We will denote by $\mathrm{gsrk}(\mathcal{H})$ the maximum m for which $\mathcal{H}$ has m independent global sections and call this number the ***global section rank*** *of $\mathcal{H}$.*

If $m = \mathrm{gsrk}(\mathcal{H})$, there are global sections $s_1, \dots s_m$ in $H^0\mathcal{H}$ which are linearly independent on $H^0_*\mathcal{O}_{\mathbf{P}^N} = k[X_0, \dots, X_N]$. Especially, if $m = \mathrm{gsrk}(\mathcal{H}) = \mathrm{rk}(\mathcal{H})$, any set of m independent global sections generate $\mathcal{H}$ outside the hypersurface zero locus of $s_1 \wedge \dots \wedge s_m$. Of course, $\mathrm{gsrk}(\mathcal{H}) \leq \mathrm{rk}(\mathcal{H})$ and equality holds if, but not "only if ", $\mathcal{H}$ is generated by global sections.

Example 2.5 *Let X be a complete intersection $(1,d)$ ($d \geq 2$) in $\mathbf{P}^3$ and let $\mathcal{H}$ be the rank 2 reflexive sheaf defined as a (suitable) extension:*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{H} \longrightarrow \mathcal{I}_X(q) \longrightarrow 0$$

for some $q \ll 0$. Then $\mathrm{gsrk}(\mathcal{I}_X(1)) = 1 = \mathrm{rk}(\mathcal{I}_X(1))$ and $\mathrm{gsrk}(\mathcal{H}(1-q)) = 2 = \mathrm{rk}(\mathcal{H}(1-q))$, but they are not generated by their global sections.

As usual, we will denote by $\hookrightarrow$ any injective map.

Lemma 2.6 *Let $\mathcal{H}$ be a torsion free sheaf on $\mathbf{P}^N$ and consider the subsheaf $\mathcal{E}$ generated by $H^0\mathcal{H}$. For every integer n, the following are equivalent:*

1. $\mathrm{gsrk}(\mathcal{H}) \geq n$;
2. $\mathrm{rk}(\mathcal{E}) \geq n$;
3. $\mathrm{rk}(\mathcal{E}_L) \geq n$ *as an $\mathcal{O}_L$-module, for a general line L in $\mathbf{P}^N$;*
4. *there is a map $\phi\colon \mathcal{O}^n_{\mathbf{P}^N} \to \mathcal{H}$ such that ϕ_L is injective for a general line L in $\mathbf{P}^N$.*

Then $\mathrm{gsrk}(\mathcal{H}) = \mathrm{rk}(\mathcal{H})$ if and only if $H^0\mathcal{H}$ generates $\mathcal{H}$ in every point of a suitable open subset U of $\mathbf{P}^N$

Proof:

1. $\Longrightarrow$ 2. Let $\mathcal{B}$ be the image of an injective map $\phi : \mathcal{O}^n_{\mathbf{P}^N} \hookrightarrow \mathcal{H}$. Then $\mathrm{rk}(\mathcal{E}) \geq \mathrm{rk}(\mathcal{B}) = n$.

2. $\Longleftrightarrow$ 3. For a general line L, the rank of $\mathcal{E}_L$ as an $\mathcal{O}_L$-module coincides with the rank of $\mathcal{E}$ as an $\mathcal{O}_{\mathbf{P}^N}$-module.

3. $\Longrightarrow$ 4. Fix a base $s_1, \ldots, s_r$ for $H^0\mathcal{H}$ as a k-vector space and consider the corresponding surjective map $\psi\colon \mathcal{O}^r_{\mathbf{P}^N} \to \mathcal{E}$. As $\mathcal{E}$ is torsion free and L is general, then $\psi_L\colon \mathcal{O}^r_L \to \mathcal{E}_L \simeq \oplus_{i=1}^m \mathcal{O}_L(a_i)$ is surjective too and $m = \mathrm{rk}(\mathcal{E}_L) \geq n$ by hypothesis. We get a map ϕ as required, for every choice of n global section of $\mathcal{H}$ such that their restriction to L are independent.

4. $\Longrightarrow$ 1. The map ϕ induces an exact sequence:

$$0 \to \mathcal{R} \to \mathcal{O}^n_{\mathbf{P}^N} \to \mathcal{E}' \to 0$$

where both $\mathcal{E}' = Im(\phi) \subseteq \mathcal{H}$ and $\mathcal{R} = Ker(\phi) \subseteq \mathcal{O}^n_{\mathbf{P}^N}$ are torsion free. Then for a general line L, we have $\mathcal{T}or^1(\mathcal{E}', \mathcal{O}_L) = 0$ and $\mathrm{rk}(\mathcal{R}) = \mathrm{rk}(\mathcal{R}_L) = 0$ so that ϕ is injective and $\mathrm{gsrk}(\mathcal{H}) \geq n$.

Finally if $n = \mathrm{rk}(\mathcal{H}) = \mathrm{gsrk}(\mathcal{H})$, then $\mathcal{H}$ and its subsheaf $\mathcal{E}$ have the same rank and so they coincide on some open, non-empty, subset U. On the converse, if $H^0\mathcal{H}$ generates $\mathcal{H}$ on U, we can take any point $x \in U$ such that $\mathcal{H}_x$ is free and $n = \mathrm{rk}(\mathcal{H})$ sections in $H^0\mathcal{H}$ defining a map $\phi : \mathcal{O}^n_{\mathbf{P}^N} \to \mathcal{H}$ which is injective in x; then $Ker(\phi)$ is a torsion subsheaf of $\mathcal{O}^n_{\mathbf{P}^N}$, that is $Ker(\phi) = 0$.
$\diamond$

Definition 2.7 *Let $\mathcal{F}$ be a rank n torsion free sheaf on $\mathbf{P}^N$. The* ***global section type*** $\mathrm{gst}(\mathcal{F})$ *is the sequence of integers $(a_1, \ldots, a_n)$ such that $a_1 \geq \cdots \geq a_n$ and for every $i = 1, \ldots, n$*

$$\mathrm{gsrk}(\mathcal{F}(-a_i - 1)) < i \quad \text{and} \quad \mathrm{gsrk}(\mathcal{F}(-a_i)) \geq i.$$

Note that $a_i \geq 0$ if and only if $\mathrm{gsrk}(\mathcal{F}) \geq i$; moreover $\mathrm{gst}(\mathcal{F}(l)) = (a_1 + l, \ldots, a_n + l)$.

Remark 2.8 *The global section rank of $\mathcal{F}$ is strictly related to some "maximal free subsheaves" of $\mathcal{F}$, in the following sense:*

$\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$ *if and only if there is a (not unique) injective map:*

$$\phi\colon \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^N}(a_i) \hookrightarrow \mathcal{F}$$

and for every injective map $f\colon \oplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \hookrightarrow \mathcal{F}$ ($\alpha_1 \geq \cdots \geq \alpha_r$), we have $r \leq n$ and $\alpha_i \leq a_i$ for every $i = 1, \ldots, r$.

Maximal free subsheaves of a sheaf $\mathcal{F}$ are studied in [7], Chapter IV. However, the two notions of maximality are slightly different and the present one is in one sense weaker and in another sense stronger than that given by [7]. It is weaker because the sheaf $\phi(\oplus_{i=1}^{n}\mathcal{O}_{\mathbf{P}^N}(a_i))$ and its direct summand $\phi(\oplus_{i=1}^{r}\mathcal{O}_{\mathbf{P}^N}(a_i))$ ($r < n$) are maximal among the free subsheaves of $\mathcal{F}$ of the same rank and not among all the subsheaves of $\mathcal{F}$ of the same rank as in [7]. It is stronger because it is stable under isomorphisms, while in [7] it is not. The following example illustrates a few differences between the two notions of maximality.

Example 2.9 *Let $\mathcal{F}$ be a non-totally split reflexive sheaf with $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$.*

1) *$\oplus_{i=1}^{n}\mathcal{O}_{\mathbf{P}^N}(a_i)$ cannot be a maximal free subsheaf of $\mathcal{F}$ in the sense of [7] because the only maximal subsheaf of $\mathcal{F}$ of rank $n = \mathrm{rk}(\mathcal{F})$ is obviously $\mathcal{F}$ itself.*

2) *every isomorphic image of $\mathcal{O}_{\mathbf{P}^N}(a_1)$ is a maximal subsheaf of $\mathcal{F}$ in the sense of [7]. On the other hand, if $a_1 < a_2$, the subsheaf $\phi(\mathcal{O}_{\mathbf{P}^N}(a_2))$ is maximal, but there are subsheaves of $\mathcal{F}$ isomorphic to $\mathcal{O}_{\mathbf{P}^N}(a_2)$ that are not, namely $f\phi(\mathcal{O}_{\mathbf{P}^N}(a_1))$, where f is any form of degree $a_2 - a_1$.*

3) *$\oplus_{i=1}^{n-1}\mathcal{O}_{\mathbf{P}^N}(a_i)$ is not always a maximal subsheaf of $\mathcal{F}$ of rank $n-1$. For instance, let $\mathcal{E}$ be a non-split bundle of rank $n-1$ with $\mathrm{gst}(\mathcal{E}) = (a_1, \ldots, a_{n-1})$, let a_n be any integer $> a_{n-1}$ and $\mathcal{F} = \mathcal{E} \oplus \mathcal{O}_{\mathbf{P}^N}(a_n)$, so that $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$. Then every subsheaf of $\mathcal{F}$ isomorphic to $\oplus_{i=1}^{n-1}\mathcal{O}_{\mathbf{P}^N}(a_i)$ is strictly contained in $\mathcal{E}$ and so it can not be a maximal free subsheaf of $\mathcal{F}$ in the sense of [7].*

Lemma 2.10 *Let $\mathcal{F}$ be a rank n reflexive sheaf on $\mathbf{P}^N$ and let $\mathcal{F}_H$ be its restriction to a general linear space $H \cong \mathbf{P}^r$ in $\mathbf{P}^N$. If $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$ and $\mathrm{gst}(\mathcal{F}_H) = (a'_1, \ldots, a'_n)$ (as a reflexive sheaf on $\mathbf{P}^r$), then:*

$$a_i \leq a'_i \quad \text{for every} \quad i = 1, \ldots n.$$

In particular, if char $k = 0$: $a_i \leq b_i$, where $(b_1, \ldots, b_n) = st(\mathcal{F}) = gst(\mathcal{F}_L)$ for a general line L in $\mathbf{P}^N$.

Proof: Consider an injective map $\phi : \oplus_{i=1}^{n}\mathcal{O}_{\mathbf{P}^N}(a_i) \hookrightarrow \mathcal{F}$. For a general linear space H we have $\mathcal{T}or^1(\mathrm{coker}(\phi), \mathcal{O}_H) = 0$ and then the the restriction

$$\phi_L : \oplus_{i=1}^{n}\mathcal{O}_H(a_i) \longrightarrow \mathcal{F}_H$$

is still injective, so that $a_i \leq a'_i$.

Moreover, for a general line L, $\mathcal{F}_L \cong \oplus_i \mathcal{O}_L(b_i)$, so that the splitting type of $\mathcal{F}$ is precisely the global section type of the restriction $\mathcal{F}_L$ to a general line L.

$\diamond$

Lemma 2.11 *Let $\mathcal{F}$ be a rank n reflexive sheaf on $\mathbf{P}^N$ and let $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$. If $\mathrm{gsrk}(\mathcal{F}) \geq c$, then:*

(i) the integers $a_1, \ldots, a_c$ are non-negative;

(ii) every $f\colon \oplus_{i=1}^{c} \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \hookrightarrow \mathcal{F}$ (where $\alpha_1 \geq \cdots \geq \alpha_c \geq 0$) factorizes through

$$\tilde{f}\colon \oplus_{i=1}^{c} \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \hookrightarrow \mathcal{G} \quad \text{and} \quad \hat{f} : \mathcal{G} \hookrightarrow \mathcal{F}$$

where $\mathcal{G}$ is a rank c reflexive sheaf and $\mathrm{coker}(\hat{f})$ is torsion free.

(iii) If $char(k) = 0$ and $\mathrm{st}(\mathcal{F}) = (b_1, \ldots, b_n)$, then $b_1, \ldots, b_c$ are non-negative; moreover if $b_{c+1}, \ldots, b_n$ are strictly negative, then $\mathrm{st}(\mathcal{G}) = (b_1, \ldots, b_c)$.

(iv) If $c = n-1$, then $\hat{f}$ can be included in the exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\hat{f}} \mathcal{F} \xrightarrow{g} \mathcal{I}_Y(q) \longrightarrow 0. \tag{4}$$

where Y is either empty or a codimension ≥ 2 subvariety of $\mathbf{P}^N$.

Proof: Part *(i)* is just the note following Definition 2.7.

(ii) Let $\mathcal{R}$ be the rank $n-c$ sheaf $\mathrm{coker}(f)$. Dualizing twice the exact sequence

$$0 \longrightarrow \oplus_{i=1}^{c} \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \xrightarrow{f} \mathcal{F} \longrightarrow \mathcal{R} \longrightarrow 0.$$

we obtain

$$0 \longrightarrow \mathcal{G} \xrightarrow{\hat{f}} \mathcal{F}^{\vee\vee} \xrightarrow{g} \mathcal{R}^{\vee\vee} \tag{5}$$

where $\mathcal{G}$ is a rank c reflexive sheaf containing $\oplus_{i=1}^{c}\mathcal{O}_{\mathbf{P}^N}(\alpha_i)$, so that $\mathrm{gsrk}(\mathcal{G}) = \mathrm{rk}(\mathcal{G}) = c$ and there is $\tilde{f}\colon \oplus_{i=1}^{c} \mathcal{O}_{\mathbf{P}^N}(\alpha_i) \hookrightarrow \mathcal{G}$. Furthermore the sheaf $\mathrm{coker}(\hat{f})$ is torsion-free because it is a subsheaf of $\mathcal{R}^{\vee\vee}$ which is reflexive.

(iii) By Lemma 2.10 and part *(i)*, we know that $b_1, \ldots, b_c \geq 0$ and also that $\beta_1 \geq \cdots \geq \beta_c \geq 0$, where $(\beta_1, \ldots, \beta_c) = st(\mathcal{G})$.

Suppose $b_{c+1}, \ldots, b_n < 0$. For a general line L:

$$\mathcal{G}_L \cong \oplus_{j=1}^{c}\mathcal{O}_L(\beta_j) \overset{\hat{f}_L}{\hookrightarrow} \mathcal{F}_L \cong \oplus_{i=1}^{n}\mathcal{O}_L(b_i).$$

So $\mathrm{Im}(\hat{f}_L)$ is a rank c subsheaf of $\oplus_{i=1}^{c}\mathcal{O}_L(b_i)$ and then the quotient $\oplus_{i=1}^{c}\mathcal{O}_L(b_i)/\mathrm{Im}(\hat{f}_L)$ is a torsion sheaf; on the other hand, it is also isomorphic to a subsheaf of $(\mathcal{R}^{\vee\vee})_L$ which is torsion-free. Thus $\mathcal{G}_L \cong \mathrm{Im}(\hat{f}_L) \cong \oplus_{i=1}^{c}\mathcal{O}_L(b_i)$.

(iv) If $n = c+1$, the sheaf $\mathcal{R}^{\vee\vee}$ given by (5) is a rank 1 reflexive sheaf, that is $\mathcal{R}^{\vee\vee} \cong \mathcal{O}_{\mathbf{P}^N}(s)$. Then $\mathrm{Im}(g)$ is a subsheaf of $\mathcal{O}_{\mathbf{P}^N}(s)$; we can write it as a suitable twist of the ideal sheaf of a subvariety Y of codimension ≥ 2 and get the exact sequence (4).

$\diamond$

3 Sharp lower bounds on c_1 and c_2

In this section $\mathcal{F}$ will always be a rank n reflexive sheaf on $\mathbf{P}^N$ with Chern classes c_i and global section type $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$. We want to state some relations between $c_1(\mathcal{F})$ and the number $\delta(\mathcal{F}) := \sum_{i=1}^{n} a_i$ and also between $c_2(\mathcal{F})$ and the number $\gamma(\mathcal{F}) := \sum_{1\leq i<j\leq n} a_i a_j$. By the definition itself of type by global sections, there is a maximal injective map $\oplus_{i=1}^{n}\mathcal{O}_{\mathbf{P}^N}(a_i) \hookrightarrow \mathcal{F}$; the integers $\delta(\mathcal{F})$ and $\gamma(\mathcal{F})$ are precisely the first and second Chern class of $\oplus\mathcal{O}_{\mathbf{P}^N}(a_i)$. When $n=1$, we assume $\gamma(\mathcal{F}) = 0$.

First of all, we collect some properties that we will use many times.

Lemma 3.1 *Let $\mathcal{F}$ be a rank n reflexive sheaf.*

(i) $c_1(\mathcal{F}(l)) - \delta(\mathcal{F}(l))$ does not depend on l that is $c_1(\mathcal{F}(l)) - \delta(\mathcal{F}(l)) = c_1 - \delta(\mathcal{F})$;

(ii) $c_2(\mathcal{F}(l)) - \gamma(\mathcal{F}(l)) = c_2 - \gamma(\mathcal{F}) + (c_1 - \delta(\mathcal{F})) \cdot (n-1) \cdot l$.

We now assume $a_n = 0$ (and then $\mathrm{gsrk}(\mathcal{F}) = n$).

(iii) There is an exact sequence:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(q) \longrightarrow 0 \tag{6}$$

where $\mathcal{G}$ is reflexive, $\mathrm{rk}(\mathcal{G}) = \mathrm{gsrk}(\mathcal{G}) = n-1$, $\mathrm{gst}(\mathcal{G}) = (a_1, \dots, a_{n-1})$, $H^0\mathcal{I}_Y(q) \neq 0$, $q \geq 0$, $H^0\mathcal{I}_Y(q-1) = 0$, $\gamma(\mathcal{G}) = \gamma(\mathcal{F})$, $\delta(\mathcal{G}) = \delta(\mathcal{F})$.

Moreover:

(iv) if $q = 0$, then Y is empty; the converse is true with the hypothesis $H^1\mathcal{G}(-q) = 0$ and in both cases $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}$.

(v) If $\mathcal{G}$ is a direct sum of line bundles and $q = 1$, then Y is a complete intersection $(1, r)$ where $r = c_2(\mathcal{F}) - \gamma(\mathcal{F}) - \delta(\mathcal{F})$ and $\mathcal{F}$ has the short free resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-r) \longrightarrow \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^N}(a_i) \oplus \mathcal{O}_{\mathbf{P}^N}(-r+1) \longrightarrow \mathcal{F} \longrightarrow 0 \tag{7}$$

Proof: Part *(i)* and part *(ii)* can be easily obtained by a straightforward computation.

To show *(iii)* we apply Lemma 2.11 to the map $\oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i) \hookrightarrow \mathcal{F}$. If $\mathrm{gst}(\mathcal{G}) = (a'_1, \dots, a'_{n-1})$ then $a'_i \geq a_i$ because there is an injective map $\oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i) \hookrightarrow \mathcal{G}$; on the other hand, $\oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a'_i) \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{F}$ implies $a'_i \leq a_i$.

Moreover we have $H^0\mathcal{I}_Y(q) \neq 0$ so that $q \geq 0$: otherwise $\mathcal{G}$ would be isomorphic to the rank n subsheaf of $\mathcal{F}$ generated by its global sections. Observe that, thanks to the assumption $a_n = 0$, we also have $h^0\mathcal{I}_Y(q-1) = 0$. Finally, the equalities on δ and γ are immediate consequence of the assumption $a_n = 0$.

For *(iv)* observe that $q = 0$ implies $Y = \emptyset$ (because $H^0\mathcal{I}_Y \neq 0$) so that the sequence (6) is exact on global sections and splits, that is $\mathcal{F} = \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}$. On the other hand, if $Y = \emptyset$ and $H^1\mathcal{G}(-q) = 0$, then $\mathcal{F}$ is the trivial extension, because $\mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^N}(q), \mathcal{G}) \cong H^1\mathcal{G}(-q)$; moreover q must be 0 because $\mathrm{gst}(\mathcal{F}) = (a_1, \dots, a_{n-1}, q) = (a_1, \dots, a_n)$ and $a_n = 0$.

Finally, in order to prove *(v)*, assume $\mathcal{G} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i)$ so that Y is a subvariety of pure codimension 2 (see Lemma 2.3); if $q = 1$, then Y is contained in a hyperplane, so that it is a complete intersection $(1, r)$, where $r = \deg_2(Y)$.

Using mapping cone on (6) and the standard free resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-r) \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-r+1) \oplus \mathcal{O}_{\mathbf{P}^N} \longrightarrow \mathcal{I}_Y(1) \longrightarrow 0$$

we get the required free resolution for $\mathcal{F}$.

◇

Remark 3.2 *The assumption $a_n = 0$ which appears in Lemma 3.1 will play a key role in the following, because very often it leads to easier computations. For instance, if $\mathcal{F}$ and $\mathcal{G}$ are as in Lemma 3.1 (iii) and $a_n = 0$, then $\delta(\mathcal{G}) = \sum_{1 \leq i \leq n-1} a_i = \sum_{1 \leq i \leq n} a_i = \delta(\mathcal{F})$ and $\gamma(\mathcal{G}) = \sum_{1 \leq i < j \leq n-1} a_i a_j = \sum_{1 \leq i < j \leq n} a_i a_j = \gamma(\mathcal{F})$: in the sequel when $a_n = 0$ we will use simply δ and γ and $\sum a_i$, $\sum a_i a_j$ for both sheaves.*

Theorem 3.3 *Let $\mathcal{F}$ be a reflexive sheaf with $\mathrm{gst}(\mathcal{F}) = (a_1, \dots, a_n)$. If $\mathrm{gsrk}(\mathcal{F}) = rk(\mathcal{F}) = n$, then:*

$$c_1(\mathcal{F}) \geq \sum_{i=1}^{n} a_i \qquad , \qquad c_2(\mathcal{F}) \geq \sum_{1 \leq i < j \leq n} a_i a_j;$$

and equality holds in either case if and only if $\mathcal{F} \cong \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^N}(a_i)$.

Proof: Thanks to Lemma 3.1 *(i)* and *(ii)*, it is sufficient to prove the statement for the minimal twist of $\mathcal{F}$ which satisfies our hypothesis: thus without lost in generality we may assume $a_n = 0$.

We proceed by induction on n. As the statement clearly holds for line bundles, suppose $n \geq 2$ and the statement true for any rank $n-1$ reflexive sheaf; thus it holds in particular for the sheaf $\mathcal{G}$ defined in Lemma 3.1 *(iii)*.

For the inequality on c_1, we just have to observe that

$$c_1(\mathcal{F}) = c_1(\mathcal{G}) + q \geq c_1(\mathcal{G}) \geq \sum_{i=1}^{n-1} a_i$$

where the last inequality is obtained applying the inductive hypothesis on $\mathcal{G}$. Using the assumption $a_n = 0$ (so that $\delta := \delta(\mathcal{F}) = \delta(\mathcal{G})$ and $\gamma := \gamma(\mathcal{F}) = \gamma(\mathcal{G})$), we have $c_1(\mathcal{F}) \geq \delta(\mathcal{F}) = \delta(\mathcal{G})$. If $c_1(\mathcal{F}) = \delta$, then also $c_1(\mathcal{G}) = \delta$ so that, by the inductive hypothesis, $\mathcal{G} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i)$ and moreover $q = 0$, so that $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}$ (see Lemma 3.1 *(iv)*).

For the second Chern class we have:

$$c_2(\mathcal{F}) = c_2(\mathcal{G}) + c_1(\mathcal{G})q + \deg_2(Y) \geq \gamma$$

where the last inequality is due to the inductive hypothesis $c_2(\mathcal{G}) \geq \gamma$.

Finally, if $c_2(\mathcal{F}) = \gamma$, we also have $c_2(\mathcal{G}) = \gamma$ (so that by induction $\mathcal{G} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i)$) and $\deg_2(Y) = 0$ (so that again $Y = \emptyset$, thanks to Lemma 2.3). Observing that in this case $H^1_* \mathcal{G} = 0$, we can conclude using lemma 3.1 *(iv)*.

$\diamond$

Remark 3.4 *The result of Theorem 3.3 on $c_1(\mathcal{F})$ holds without the hypothesis* $\mathrm{gsrk}(\mathcal{F}) = rk(\mathcal{F}) = n$, *because $c_1(\mathcal{F}) - \sum a_i$ is invariant by twist, while this hypothesis is necessary for the result about $c_2(\mathcal{F})$ because $c_2 - \sum a_i a_j$ is not invariant by twist (Lemma 3.1, (ii)).*

The following example shows that for every choice of non-negative integers $a_1, \ldots, a_n$ and s, there are rank n reflexive sheaves $\mathcal{F}$ such that $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$ and $c_1(\mathcal{F}) = \sum a_i + s$; in other words, $c_1(\mathcal{F})$ can in fact reach every value above the minimal one $\delta(\mathcal{F})$ given by the previous results.

Example 3.5 *Let $a_1 \geq \cdots \geq a_n$ and s be non-negative integers. We define $p \geq a_1 - a_2 + s$ and let Y be a complete intersection (s, p) in $\mathbf{P}^N$.*

As $\omega_Y \cong \mathcal{O}_Y(s + p - N - 1)$, then $\omega_Y(N + 1 - s - a_1 + a_2) \cong \mathcal{O}_Y(p - a_1 + a_2)$ has a section which generates it almost everywhere. Such a section gives an extension:

$$0 \to \mathcal{O}_{\mathbf{P}^N}(a_2) \to \mathcal{G} \to \mathcal{I}_Y(s + a_1) \to 0$$

where $\mathcal{G}$ is a rank 2 reflexive sheaf with first Chern class $c_1(\mathcal{G}) = a_1 + a_2 + s$; moreover by the hypothesis on p we have $H^0\mathcal{I}_Y(s-1) = 0$ and $H^0\mathcal{I}_Y(s + a_1 - a_2 - 1) = H^0\mathcal{I}_Y(s) \otimes H^0\mathcal{O}_{\mathbf{P}^N}(a_1 - a_2 - 1)$, so that $\mathrm{gst}(\mathcal{G}) = (a_1, a_2)$.

Finally, the rank n reflexive sheaf $\mathcal{F} = \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}(a_3) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^N}(a_n)$ has first Chern class $c_1(\mathcal{F}) = \sum a_i + s$ and global section type $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$.

Note that such a sheaf has the short free resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(a - s) \longrightarrow \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^N}(a_i) \oplus \mathcal{O}_{\mathbf{P}^N}(a) \longrightarrow \mathcal{F} \longrightarrow 0 \tag{8}$$

with $a = a_1 - p + s$.

All the sheaves obtained in Example 3.5 are of a very special type: they have homological dimension ≤ 1 and a short free resolution. Now we will show that in fact every sheaf for which $c_1(\mathcal{F}) = \delta(\mathcal{F}) + 1$ has the same nice properties. Note that we can obtain reflexive sheaves $\mathcal{F}$ with a resolution of the type (8) and $\mathrm{gst}(\mathcal{F}) = (a_1, \ldots, a_n)$ for every possible integer $a \leq a_2$.

Corollary 3.6 *Let $\mathcal{F}$ be a reflexive sheaf. If $\mathcal{F}$ is not a sum of line bundles, then*

$$c_1(\mathcal{F}) \geq \sum a_i + 1$$

and equality holds if and only if $\mathrm{hd}(\mathcal{F}) \leq 1$ *and $\mathcal{F}$ has a free resolution (7).*

Proof: We have to show that $c_1(\mathcal{F}) - \delta(\mathcal{F}) \geq 1$. As $c_1(\mathcal{F}) - \delta(\mathcal{F})$ does not change up a twist, we may assume $a_n = 0$: so the inequality follows from Theorem 3.3.

Moreover $a_n = 0$ also allows us to use the exact sequence (6). Assume that $c_1(\mathcal{F}) = \delta(\mathcal{F}) + 1$. If $q = 0$, then $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}$ (Lemma 3.1, *(iv)*) and we conclude by induction on the rank. If $q \neq 0$, then $q = 1$ and we have $c_1(\mathcal{G}) = c_1(\mathcal{F}) - q = \delta(\mathcal{F}) = \delta(\mathcal{G})$, so that $\mathcal{G} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbf{P}^N}(a_i)$ (see Theorem 3.3) and we conclude thanks to Lemma 3.1 *(v)*.
$\diamond$

For what concerns the second Chern class, not all values above the minimal one given in Theorem 3.3 can in fact be realized, at least for general $a_1, \ldots, a_n$. For instance, if $a_2 > 0$, any integer in the interval $[\gamma(\mathcal{F}) + 1\ ,\ \gamma(\mathcal{F}) + \delta(\mathcal{F}) - a_2]$ is not allowed. More precisely:

Theorem 3.7 *Let $\mathcal{F}$ be rank $n \geq 2$ reflexive sheaf on $\mathbf{P}^N$ with Chern classes c_i and global section type $(a_1, \ldots, a_n)$. Then:*

$$c_2 \geq \sum_{1 \leq i < j \leq n} a_i a_j + \left(c_1 - \sum_{1 \leq i \leq n} a_i \right) \left(\sum_{1 \leq i \leq n} a_i + 1 - a_2 \right). \tag{9}$$

Proof: First of all observe that equality holds in (9) if $\mathcal{F} \cong \oplus \mathcal{O}_{\mathbf{P}^N}(a_i)$ is a sum of line bundles, because in this case $c_1(\mathcal{F}) = \sum a_i$ and $c_2(\mathcal{F}) = \sum a_i a_j$. So, assume that $\mathcal{F}$ is not split.

We have to show that the integer $\Delta(\mathcal{F}) = c_2 - \gamma - (c_1 - \delta)(\delta + 1 - a_2)$ is non-negative. Using Lemma 3.1 *(i)* and *(ii)*, we can see that $\Delta(\mathcal{F})$ is invariant under twist, that is $\Delta(\mathcal{F}(l)) = \Delta(\mathcal{F})$ for every $l \in \mathbb{Z}$. Thus, without lost in generality, we may assume $a_n = 0$.

Let $\mathcal{G}$ and Y be as in Lemma 3.1 *(iii)* and denote by d_i the Chern classes of $\mathcal{G}$. By the exact sequence (6) and Lemma 2.3, we get:

$$c_2 = d_2 + (c_1 - d_1)d_1 + \deg_2(Y) \geq \gamma + (d_1 - \delta)(\delta + 1 - a_2) + (c_1 - d_1)d_1 + \deg_2(Y).$$

If $\mathrm{rk}(\mathcal{F}) = 2$, then $\mathcal{G} \cong \mathcal{O}_{\mathbf{P}^N}(a_1)$ so that the exact sequence (6) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(a_1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(c_1 - a_1) \longrightarrow 0$$

and Y is a subvariety of pure codimension 2 (see Lemma 2.3) not contained in hypersurfaces of degree $c_1 - a_1 - 1$, because on the contrary $a_2 \geq 1$ against the assumption. Moreover $d_1 = \delta = a_1$, $d_2 = \gamma = 0$. So it will be sufficient to prove that $(c_1 - d_1)d_1 + \deg_2(Y) \geq (c_1 - a_1)(a_1 + 1)$ that is $\deg_2(Y) \geq (c_1 - d_1)$. This is true because every subvariety Y of pure codimension 2 and degree s is always contained in some degree s hypersurfaces (for instance cones).

If $\mathrm{rk}(\mathcal{F}) \geq 3$, we can proceed by induction on the rank and assume that (9) holds for the reflexive sheaf $\mathcal{G}$. Thanks to the equality:

$$\Delta(\mathcal{F}) = \Delta(\mathcal{G}) + \deg_2(Y) - (c_1 - d_1)(\delta + 1 - a_2 - d_1)$$

and the inductive hypothesis, it is sufficient to prove that:

$$\deg_2(Y) \geq (c_1 - d_1)(\delta + 1 - a_2 - d_1). \tag{10}$$

We know that $c_1 \geq d_1 \geq \delta$ and $a_2 \geq a_n = 0$ (see Theorem 3.3); then (10) clearly holds if either $a_2 > 0$ or $d_1 > \delta$.

The only case left to consider is $a_2 = d_1 - \delta = 0$; in this case $\mathcal{G} \cong \mathcal{O}_{\mathbf{P}^N}(a_1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$ (again by Theorem 3.3) so that $d_1 = a_1$, Y is a subvariety of pure codimension 2 (see Lemma 2.3), $H^0\mathcal{I}_Y(c_1 - a_1 - 1) = 0$ and the exact sequence (6) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(a_1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(c_1 - a_1) \longrightarrow 0.$$

Thus we can now conclude as in the rank 2 case.

◇

Now we list a few remarkable consequences of Corollary 3.6 and Theorem 3.7. Note that the integer $\sum a_i - a_2 + 1 = \sum_{i\neq 2} a_i + 1$ is strictly positive when $\mathrm{gsrk}(\mathcal{F}) = n$.

Corollary 3.8 *Let $\mathcal{F}$ be a rank n reflexive sheaf and let $\alpha_1, \ldots, \alpha_n$ be integers such that $\mathcal{F}$ has a free subsheaf isomorphic to $\oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^N}(\alpha_i)$: for instance $(\alpha, \ldots, \alpha_n) = \mathrm{gst}(\mathcal{F})$. If $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, then:*

$$c_1 \geq \sum_{1\leq i\leq n} \alpha_i \quad , \quad c_2 \geq \sum_{1\leq i<j\leq n} \alpha_i\alpha_j.$$

Moreover equality on c_1 holds if and only if $\mathcal{F}$ is $\oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^N}(\alpha_i)$; equality on c_2 holds if and only if $\mathcal{F}$ is either $\oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^N}(\alpha_i)$ or $\mathcal{O}_{\mathbf{P}^N}^{n-1} \oplus \mathcal{O}_{\mathbf{P}^N}(a_1)$ for some $a_1 > \alpha_1$.

If $\mathcal{F}$ is not a direct sum of line bundles then

$$c_2(\mathcal{F}) \geq \sum_{1\leq i<j\leq n} \alpha_i\alpha_j + \sum_{i=1}^n \alpha_i + 1 - \alpha_2 \tag{11}$$

and moreover

$$c_2(\mathcal{F}) \geq \sum_{1\leq i<j\leq n} \alpha_i\alpha_j + 2(\sum_{i=1}^n \alpha_i + 1 - \alpha_2) \tag{12}$$

unless $\mathcal{F}$ has the short free resolution given in (7).

Corollary 3.9 *Let $\mathcal{F}$ be a reflexive sheaf such that $\mathrm{gsrk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F})$. Then*

$$c_1(\mathcal{F}) \geq 0 \quad , \quad c_2(\mathcal{F}) \geq 0$$

and either equality holds only if $\mathcal{F} = \mathcal{O}_{\mathbf{P}^N}^{n-1} \oplus \mathcal{O}_{\mathbf{P}^N}(c_1)$.

Example 3.10 *Sharp cases for the lower bounds on c_2 given by the previous results when $(\alpha_1, \ldots, \alpha_n) = (a_1, \ldots, a_n) = \mathrm{gst}(\mathcal{F})$ can be found in Example 3.5 for special values of the parameters.*

For every $a_1 \geq \cdots \geq a_n \geq 0$, we can choose $s = 1$ and $p = a_1 - a_2 + 1$ and get a sheaf $\mathcal{F}$ such that $c_1 = \delta + 1$ and $c_2 = \gamma + (\delta - a_2 + 1)$.

For what concerns sheaves with homological dimension ≥ 2 (and then without the free resolution (7)) a sharp case is given by the rank 2 reflexive sheaf $\mathcal{F}$ that we will consider in Example 5.6: its global section type is $(a_1 = 0, a_2 = 0)$ and its first and second Chern classes are $c_1 = 2$ and $c_2 = 2$. Using such a sheaf $\mathcal{F}$ we can also find reflexive sheaves of any rank $n > 2$ on $\mathbf{P}^4$ realizing the equality in (12): for instance $\mathcal{F}(l) \oplus \mathcal{O}_{\mathbf{P}^4}(a_3) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^4}(a_n)$, where $l \geq a_3 \geq \cdots \geq a_n \geq 0$.

We conclude this section showing that in the previous results we cannot simply avoid the assumption on the global section rank in order to get the positivity of c_1 and c_2.

Example 3.11 *Let q be an integer, $q >> 0$, and $\mathcal{E}$ be the rank 2 bundle on $\mathbf{P}^3$ defined as an extension*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y(-2q+4) \longrightarrow 0$$

where Y is a $(-2q)$-subcanonical double structure on a line obtained by Ferrand construction (see [4], Theorem 1.5).

We have

$$c_1(\mathcal{E}) = -2q + 4 \qquad c_2(\mathcal{E}) = 2 \qquad h^0\mathcal{E} \neq 0.$$

Then for every integer t, $0 < t < q - 2$, the global section rank is strictly lower than the rank (in fact $\mathrm{gsrk}(\mathcal{E}(t)) = \mathrm{rk}(\mathcal{E}(t)) - 1$) and both $c_1(\mathcal{E}(t))$ and $c_2(\mathcal{E}(t))$ are strictly negative.

4 Special splitting types

As we have seen in the previous section, not all sheaves whose global section rank is lower than the rank have positive c_1 and c_2, not even vector bundles. In this section we assume something weaker about the global section rank (namely $\mathrm{gsrk}(\mathcal{F}) \geq \mathrm{rk}(\mathcal{F})-1$), while we introduce balancing hypothesis on the splitting type.

Proposition 4.1 *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^N$ such that $\mathrm{rk}(\mathcal{F}) = n$, $\mathrm{gsrk}(\mathcal{F}) = n-1$, $c_1 \leq 0$ and $\mathrm{st}(\mathcal{F}) = (0,\dots,0,c_1)$. Then:*

$$c_2(\mathcal{F}) \geq 0.$$

Moreover $c_2(\mathcal{F}) = 0$ if and only if $\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^N}(c_1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-1}$.

Proof: Let $\mathcal{G}$ and Y be as in Lemma 2.11: under the present assumption on $\mathcal{F}$, we have $\mathrm{rk}(\mathcal{G}) = \mathrm{gsrk}(\mathcal{G}) = n-1$ and $\mathrm{st}(\mathcal{G}) = (0,\dots,0)$ so that $c_1(\mathcal{G}) = 0$. The only sheaf of such a type is $\mathcal{G} \cong \mathcal{O}_{\mathbf{P}^N}^{n-1}$ (see Corollary 3.9) and so by (4) we get $c_2(\mathcal{F}) = c_2(\mathcal{I}_Y(c_1)) = \deg_2(Y) \geq 0$.

Moreover Y has pure codimension 2 or it is empty (see Lemma 2.3); then $c_2 = 0$ if and only if $Y = \emptyset$ and $\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^N}(c_1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-1}$.
$\diamond$

Remark 4.2 *Most properties concerning the sheaf $\mathcal{F}$ that appear in the previous section are in general "stable by positive twist"; for instance for every $l \geq 0$, if $c_1(\mathcal{F}) \geq 0$ and $c_2(\mathcal{F}) \geq 0$, then also $c_1(\mathcal{F}(l)) \geq 0$ and $c_2(\mathcal{F}(l)) \geq 0$ and if $\mathrm{gsrk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F})$ then also $\mathrm{gsrk}(\mathcal{F}(l)) = \mathrm{rk}(\mathcal{F}(l))$. On the contrary neither the hypothesis nor the thesis that appear in Proposition 4.1 are "stable", as shown by the following example.*

Example 4.3 *Let Y be a line in $\mathbf{P}^3$. For any integer $c \leq -2$, $\omega_Y(4-c) \cong \mathcal{O}_Y(2-c)$ has a section which generates it almost everywhere. Therefore there is an extension (see [3], proof of 1.1)*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y(c) \longrightarrow 0$$

which defines the rank 2 reflexive sheaf $\mathcal{E}$. This sheaf satisfies the hypothesis of Proposition 4.1 that is $\mathrm{gsrk}(\mathcal{E}) = 1 = \mathrm{rk}(\mathcal{E})-1$, $c_1(\mathcal{E}) = c < 0$, $\mathrm{st}(\mathcal{E}) = (0,c_1)$; as a consequence, the second Chern class $c_2(\mathcal{E})$ is positive (in fact $c_2(\mathcal{E}) = \deg_2 Y = 1$). On the other hand, $\mathcal{E}(1)$ has a negative first Chern class, but it does not satisfy the hypothesis of Proposition 4.1 about the splitting type, which is in fact $(1, c+1)$ instead of $(0, c+2)$. If we compute its second Chern class $c_2(\mathcal{E}(1)) = c+2$, we can see that $c_2(\mathcal{E}(1))$ is strictly negative when $c < -2$ and moreover that $c_2(\mathcal{E}(1)) = 0$ when $c = -2$ although $\mathcal{E}(1)$ is not a split bundle.

Proposition 4.4 *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^N$ with Chern classes c_i, such that $\mathrm{rk}(\mathcal{F}) = n$, $\mathrm{gsrk}(\mathcal{F}) = n-1$, $c_1 \leq 0$ and $\mathrm{st}(\mathcal{F}) = (1,0,\dots,0,c_1-1)$. Then $c_2 \geq c_1 - 1$. Moreover*

(i) $c_2 = c_1 - 1$ *if and only if* $\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^N}(1) \oplus \mathcal{O}_{\mathbf{P}^N}(c_1-1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$.

(ii) *If* $H^0\mathcal{F}(-1) \neq 0$, *then* $c_2 = c_1$ *if and only if* $\mathcal{F}$ *has the following free resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(c_1-3) \longrightarrow \mathcal{O}_{\mathbf{P}^N}(c_1-2)^2 \oplus \mathcal{O}_{\mathbf{P}^N}(1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{13}$$

(iii) *If* $H^0\mathcal{F}(-1) = 0$ *and furthermore* $H^1\mathcal{F}_{H_r}(-1) = 0$ *for every general linear subspace* $H_r \cong \mathbf{P}^r$ *in* $\mathbf{P}^N$ *($r \geq 3$), then* $c_2 = c_1$ *if and only if* $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}(c_1-1)$ *and* $\mathcal{G}$ *has the free resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^N}^n \longrightarrow \mathcal{G} \longrightarrow 0 \tag{14}$$

Proof: Let $\mathcal{G}$ be the reflexive sheaf constructed in Lemma 2.11 (4); denote by d_i its Chern classes. We have $\mathrm{rk}(\mathcal{G}) = \mathrm{gsrk}(\mathcal{G}) = n-1$ (so that $d_2 \geq 0$: see Corollary 3.9) and moreover $\mathrm{st}(\mathcal{G}) = (1, 0, \dots, 0)$ (so that $d_1 = 1$). Using the exact sequence (4) we obtain:

$$c_2 = d_2 + d_1(c_1 - d_1) + \deg_2(Y) = d_2 + c_1 - 1 + deg_2(Y) \geq c_1 - 1. \tag{15}$$

Note that $c_1 - d_1 = c_1 - 1 < 0$; then $H^0\mathcal{I}_Y(c_1 - 1) = 0$ so that $H^0\mathcal{F}(-1) \cong H^0\mathcal{G}(-1)$.

(i) The equality $c_2 = c_1 - 1$ can hold only if $d_2 = 0$ so that $\mathcal{G} \cong \mathcal{O}_{\mathbf{P}^N}(1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$ (Corollary 3.9); moreover Y is a pure codimension 2 subvariety of degree 0 that is $Y = \emptyset$ and $\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^N}(1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2} \oplus \mathcal{O}_{\mathbf{P}^N}(c_1 - 1)$.

The next value $c_2 = c_1$ can be realized if and only if either $d_2 = 0$ and $\deg_2(Y) = 1$ or $d_2 = 1$ and $\deg_2(Y) = 0$.

(ii) In the first case $\mathcal{G} \cong \mathcal{O}_{\mathbf{P}^N}(1) \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$, Y is a pure codimension 2 subvariety of degree 1, that is a complete intersection $(1, 1)$. Using mapping cone on (4) and the canonical free resolution of $\mathcal{I}_Y(c_1 - 1)$, we get (13).

(iii) In the other case, namely if $d_2 = 1$ and $\deg_2(Y) = 0$, we find $\mathrm{gst}(\mathcal{G}) = (0, \dots, 0)$ and then $\mathcal{G}$ has the free resolution (14) (see Corollary 3.8) and so $H^1_*\mathcal{G} = 0$. To complete the proof we just have to show that Y is empty.

If not, let r be the codimension of Y, $3 \leq r \leq N$; the restriction $Y' = Y \cap H_r$ to a sufficiently general linear subspace H_r is a finite set of points, whose degree $\delta = \deg(Y)$ is given by $h^0\mathcal{O}_{Y'}(c_1 - 2) = h^1\mathcal{I}_{Y'}(c_1 - 2)$. On the other hand $h^1\mathcal{I}_{Y'}(c_1 - 2) \leq h^1\mathcal{F}_{H_r}(-1) + h^2\mathcal{G}(-1) = 0$. Then Y is empty and $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}(c_1 - 1)$ (see Lemma 3.1, *(iv)*).

$\diamond$

5 Positivity and border cases for c_3

In the present, last section we want to investigate general conditions that ensure the positivity of the third Chern class c_3 of a reflexive sheaf $\mathcal{F}$ of any rank.

To this aim it will be useful to have a deeper knowledge about the singular locus of $\mathcal{F}$ or, more generally, about the set of points where it is not freely generated.

Proposition 5.1 *Let $\mathcal{F}$ be a rank n reflexive sheaf on $\mathbf{P}^N$ such that* $\mathrm{hd}(\mathcal{F}) \leq 1$, $\mathrm{hd}(\mathcal{F}^\vee) \leq 1$.

Then the singular locus $\mathcal{S}$ of $\mathcal{F}$ is either a codimension 3 closed subset or it is empty (that is $\mathcal{F}$ is a vector bundle).

Proof: Of course there is nothing to prove if $N \leq 3$. Moreover it is equivalent to prove the statement for $\mathcal{F}$ or for its dual $\mathcal{F}^\vee$ or for some of their twists $\mathcal{F}(k)$ or $\mathcal{F}^\vee(k)$ ($k \in \mathbb{Z}$).

Assume $\mathrm{codim}(\mathcal{S}) \geq 4$ and consider a locally free resolution of $\mathcal{F}$:

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{16}$$

We may assume that $\mathcal{E}_0$ is a direct sum of line bundles. Dualizing, we get:

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{E}_0^\vee \longrightarrow \mathcal{E}_1^\vee \longrightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) \longrightarrow 0$$

which splits in two short exact sequences:

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{E}_0^\vee \longrightarrow \mathcal{L} \longrightarrow 0 \tag{17}$$

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}_1^\vee \longrightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) \longrightarrow 0 \tag{18}$$

Observe that when the singular locus $\mathcal{S}$ is a finite set of points, then $h^i(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})) = 0$ for every $i \geq 1$ because the support of $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})$ is contained in $\mathcal{S}$. In this case using (16),(17),(18) and duality on $\mathcal{E}_1$ we find:

$$H^0\mathcal{E}_1^\vee \longrightarrow H^0\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) \longrightarrow H^1\mathcal{L} \longrightarrow H^1\mathcal{E}_1^\vee \longrightarrow 0 \tag{19}$$

and moreover

$$h^1\mathcal{L} = h^2\mathcal{F}^\vee \text{ and } h^1\mathcal{E}_1^\vee = h^{N-1}\mathcal{E}_1(-N-1) = h^{N-2}\mathcal{F}(-N-1).$$

Then $h^2\mathcal{F}^\vee \geq h^{N-2}\mathcal{F}(-N-1)$.

We split the proof in three steps.

- First we prove the statement in $\mathbf{P}^4$. By hypothesis, $\mathcal{S}$ is a 0-dimensional set and so $h^2\mathcal{F}^\vee \geq h^2\mathcal{F}(-5)$. Since the same inequality holds for every twist of $\mathcal{F}$ and $\mathcal{F}^\vee$, then equality holds, so that $h^1\mathcal{L} = h^1\mathcal{E}_1^\vee$. On the other hand, up a suitable twist of $\mathcal{F}$, we have $H^0\mathcal{E}_1^\vee = 0$ and then $H^0\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^4}) = 0$, so that $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) = 0$ (because it is a constant sheaf supported on a finite set of points) and $\mathcal{L} = \mathcal{E}_1^\vee$. Thus we have:

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{E}_0^\vee \longrightarrow \mathcal{E}_1^\vee \longrightarrow 0$$

  and $\mathcal{F}^\vee$ is locally free.

- Now we prove the statement for every $N \geq 5$ assuming that $\mathcal{S}$ is a finite set of points. As above, $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})$ has a finite support and, using again (16),(17),(18) and duality on $\mathcal{E}_1$ we get:

$$\begin{aligned} h^2\mathcal{F} = h^3\mathcal{E}_1 = h^{N-3}(\mathcal{E}_1^\vee(-N-1)) = & \text{ (note that } N-3 \geq 2\,) \\ = & \; h^{N-3}\mathcal{L}(-N-1) \quad = h^{N-2}\mathcal{F}^\vee(-N-1) \end{aligned}$$

  Since the same equality holds for every twist of $\mathcal{F}$ and $\mathcal{F}^\vee$, then $H^2_*(\mathcal{F})$ and $H^2_*(\mathcal{F}^\vee)$ are finite modules. Then for every $t \gg 0$, we have $h^0\mathcal{E}_1^\vee(-t) = 0$ and $h^1\mathcal{L}(-t) = h^2\mathcal{F}^\vee(-t) = 0$, so that by the cohomology exact sequence of (18) we obtain that $H^0\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})(-t) = 0$. This implies $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) = 0$, because it is a constant sheaf supported on a finite set of points; from this, as above, we deduce that $\mathcal{F}$ is locally free.

- Finally we consider the general case and proceed by induction on N. Let $N \geq 5$ and assume the statement true for $N-1$. If H is a general hyperplane, we can apply Lemma 2.1 and see that the restriction $\mathcal{F}_H$ is a reflexive sheaf on H, whose singular locus $\mathcal{S}(\mathcal{F}_H) = \mathcal{S} \cap H$ has codimension ≥ 4 in H; moreover short locally free resolutions of $\mathcal{F}$ and $\mathcal{F}^\vee$ restrict to short locally free resolution of $\mathcal{F}_H$ and $(\mathcal{F}^\vee)_H \cong (\mathcal{F}_H)^\vee$ ((Lemma 2.1 *(iii)*). Thus $\mathcal{F}_H$ satisfies the same condition as $\mathcal{F}$. By the inductive hypothesis, $\mathcal{F}_H$ is locally free on H, namely $\mathcal{S}(\mathcal{F}_H) = \mathcal{S} \cap H = \emptyset$ (Lemma 2.1 *(iv)*). This can happen only if $\mathcal{S}$ is at most a finite set of points and we conclude thanks to the previous item.

$\diamond$

Theorem 5.2 *Let $\mathcal{F}$ be a rank n reflexive sheaf on $\mathbf{P}^N$ generated by global sections outside a closed subset of codimension ≥ 3. Then:*

$$c_3(\mathcal{F}) \geq 0$$

and equality $c_3(\mathcal{F}) = 0$ can hold only if $N \geq 3$ and $\mathcal{F}_H$ is a vector bundle for every general linear subspace $H \cong \mathbf{P}^3$ in $\mathbf{P}^N$.

If moreover $\mathrm{hd}(\mathcal{F}) \leq 1$ and $\mathrm{hd}(\mathcal{F}^\vee) \leq 1$, then $c_3(\mathcal{F}) = 0$ only if $\mathcal{F}$ is a vector bundle having a direct summand $\mathcal{O}^r_{\mathbf{P}^N}$, for some $r \geq n-2-h^1\mathcal{F}(-c_1)$.

Proof: Assume that $\mathcal{F}$ is not a free bundle. Then $n-1$ general sections of $\mathcal{F}$ degenerate on a generically smooth codimension 2 subvariety Y given by the exact sequence:

$$0 \longrightarrow \mathcal{O}^{n-1}_{\mathbf{P}^N} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(c_1) \longrightarrow 0 \tag{20}$$

(see Lemma 2.2 and [9], §2); thanks to Lemma 2.3, we can see that Y has no embedded or isolated components of codimension ≥ 3.

If we apply the functor $\mathcal{H}om(\cdot, \mathcal{O}_{\mathbf{P}^N})$ to the exact sequence (20), we find:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-c_1) \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}^{n-1}_{\mathbf{P}^N} \longrightarrow \mathcal{E}xt^1(\mathcal{I}_Y(c_1), \mathcal{O}_{\mathbf{P}^N}) \longrightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N}) \longrightarrow 0 \tag{21}$$

where $\mathcal{E}xt^1(\mathcal{I}_Y(c_1), \mathcal{O}_{\mathbf{P}^N}) \cong \omega_Y(N+1-c_1)$ (see [2] Ch. III, Proposition 7.5) and $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbf{P}^N})$ is supported on $\mathcal{S}$.

Then $\mathcal{O}_{\mathbf{P}^N}^{n-1} \longrightarrow \omega_Y(N+1-c_1)$ is surjective outside $\mathcal{S}$ that is $\omega_Y(N+1-c_1)$ is generated almost everywhere by its global sections; as $\omega_Y(N+1-c_1)$ has rank 1, it is in fact generated almost everywhere by just one of them. Such a section and the isomorphisms:

$$H^0\omega_Y(N+1-c_1) \cong \mathrm{Hom}_Y(\mathcal{O}_Y, \omega_Y(N+1-c_1)) \cong$$
$$\cong \mathrm{Ext}^2_{\mathbf{P}^N}(\mathcal{O}_Y, \mathcal{O}_{\mathbf{P}^N}(-c_1)) \cong \mathrm{Ext}^1_{\mathbf{P}^N}(\mathcal{I}_Y, \mathcal{O}_{\mathbf{P}^N}(-c_1))$$

(see [2] III, Lemma 7.4), give an extension:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y(c_1) \longrightarrow 0 \tag{22}$$

where $\mathcal{E}$ is a rank 2 reflexive sheaf. Using multiplicativity of Chern classes in (20) and (22), we get

$$c_3(\mathcal{F}) = c_3(\mathcal{I}_Y(c_1)) = c_3(\mathcal{E}) = c_3(\mathcal{E}_H) \geq 0.$$

for every general $H \cong \mathbf{P}^3$ in $\mathbf{P}^N$ (see Lemma 2.1 *(v)* and [3], Theorem 4.1). Note that our hypothesis on $\mathcal{F}$ also holds for every restriction of $\mathcal{F}$ to a general linear subspace.

Finally, $c_3(\mathcal{F}) = 0$ if and only if $c_3(\mathcal{E}_H) = 0$, that is $\mathcal{E}_H$ is a vector bundle (see again [3], Theorem 4.1). The curve $C = Y \cap H$ is (c_1-4)-subcanonical that is $\omega_C(4-c_1) \cong \mathcal{O}_C$ and a section which generates it almost everywhere in fact generates it; in the exact sequence (21) the map $\mathcal{O}_{\mathbf{P}^3}^{n-1} \longrightarrow \mathcal{E}xt^1(\mathcal{I}_C(c_1), \mathcal{O}_{\mathbf{P}^3}) \cong \mathcal{O}_C$ is surjective so that $\mathcal{E}xt^1(\mathcal{F}_H, \mathcal{O}_{\mathbf{P}^3}) = 0$ that is $\mathcal{F}_H$ is locally free.

If moreover $\mathrm{hd}(\mathcal{F}) \leq 1$ and $\mathrm{hd}(\mathcal{F}^\vee) \leq 1$, by Proposition 5.1 we can conclude that $\mathcal{F}$ is a vector bundle too.

The exact sequence (21) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-c_1) \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_{\mathbf{P}^N}^{n-1} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

By a suitable change of base we can assume that $r \geq (n-2-h^0\mathcal{O}_Y)$ copies of $\mathcal{O}_{\mathbf{P}^N}$ are in fact contained in the kernel of the last map and $\mathcal{F}$ contains a direct summand $\mathcal{O}_{\mathbf{P}^N}^r$.

$\diamond$

Corollary 5.3 *Let $\mathcal{F}$ be a rank n reflexive sheaf on $\mathbf{P}^N$ ($N \geq 4$) such that $\mathrm{hd}(\mathcal{F}) \leq 1$ and $\mathrm{hd}(\mathcal{F}^\vee) \leq 1$. If $\mathcal{F}$ is generated by global sections, then:*

$$c_3(\mathcal{F}) = 0 \Longleftrightarrow \mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}, \ \textit{where } \mathcal{G} \textit{ is a rank 2 vector bundle.}$$

Proof: If $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$ and $\mathcal{G}$ is a rank 2 vector bundle, then $c_3(\mathcal{F}) = c_3(\mathcal{G}) = 0$. The converse is a straightforward consequence of the previous result. In fact for $n-1$ general global sections of $\mathcal{F}$, the zero locus Y is a codimension 2 smooth subvariety in $\mathbf{P}^N$; as $N \geq 4$, Y must be irreducible so that $h^1\mathcal{F}(-c_1) = h^1\mathcal{I}_Y = h^0\mathcal{O}_Y = 1$.

$\diamond$

Corollary 5.4 *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on $\mathbf{P}^N$ with $\mathrm{hd}(\mathcal{F}) \leq 1$. Then:*

$$c_3 = 0 \Longleftrightarrow \mathcal{F} \textit{ is a vector bundle.}$$

Proof: For rank 2 reflexive sheaves, duality $\mathcal{F}^\vee \cong \mathcal{F}(-c_1)$ implies $\mathrm{hd}(\mathcal{F}) = \mathrm{hd}(\mathcal{F}^\vee)$; moreover the third Chern class is invariant under twist. Thus we can apply the previous result to a suitable twist of $\mathcal{F}$ and conclude.

$\diamond$

Note that in the previous results we cannot simply avoid either hypothesis on the homological dimension or on global sections of $\mathcal{F}$, because in fact not every reflexive sheaf whose third Chern class vanish is a vector bundle.

Example 5.5 *Let $\mathcal{F}$ be a rank 2 reflexive sheaf with $c_2 = r$ and $c_3 = r^2$ (for instance we can find such a sheaf for special values $n = 2$ and $a_1 = a_2 = 0$ in (7)). Then $\mathcal{F} \oplus \mathcal{O}_{\mathbf{P}^N}(-r)$ is not a bundle even if $c_3 = 0$.*

Example 5.6 *Let Y be the union of two general 2-planes in $\mathbf{P}^4$ with only one common point Q. A general non-zero global section of the sheaf $\omega_Y(3)$ generates it at every point $P \in Y$ except at the point Q. Using such a section we can define an extension:*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y(2) \longrightarrow 0$$

where $\mathcal{F}$ is a rank 2 reflexive sheaf with $c_1 = 2$ and $c_2 = 2$. If we cut by a general hyperplane H, we find that Y_H is the disjoint union of two lines in $H = \mathbf{P}^3$ and that $\mathcal{F}_H(-1)$ is a null correlation bundle. Then $c_3(\mathcal{F}) = c_3(\mathcal{F}_H) = 0$ even if $\mathcal{F}$ is not locally free.

Note that in fact Y is not locally Cohen-Macaulay (at the point Q), while its general hyperplane section is. If $k \gg 0$, a general global section of $\mathcal{F}(k)$ degenerates on an integral surface which is not locally Cohen-Macaulay, while its general section is a subcanonical curve in $\mathbf{P}^3$.

We conclude by showing that we can not expect to control the positivity of the i-th Chern class for every reflexive sheaf when $i \geq 4$, even if i is lower than the rank. In fact for every pair (n, i) (both ≥ 4) there are rank n reflexive sheaves on $\mathbf{P}^N$, generated by global sections, having negative c_i. In the following example we construct sheaves of this type; it is not difficult to generalize the construction in order to obtain for every $t \in \mathbb{N}$ reflexive sheaves $\mathcal{F}$ with the same properties and such that moreover $\mathcal{F}(-t)$ is generated by global sections.

Example 5.7 *Let $\mathcal{G}'$ be any rank 2 reflexive sheaf on $\mathbf{P}^N$ with third Chern class strictly positive. If $l \gg 0$, the sheaf $\mathcal{G} = \mathcal{G}'(l)$ is generated by global sections, its first three Chern classes are positive, while the forth and following ones with even indexes are negative.*
The sheaf $\mathcal{F} = \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}^{n-2}$ is reflexive too, it is generated by global sections and it has the same Chern classes as $\mathcal{G}$, so the "even" ones are negative from the fourth on.

For the "odd" Chern classes, we can start from the rank 3 reflexive sheaf $\mathcal{G}'_1 = \mathcal{G} \oplus \mathcal{O}_{\mathbf{P}^N}(a)$ (where $\mathcal{G}$ is as above and $a \gg 0$) such that $c_4(\mathcal{G}'_1) > 0$. Again, if $l \gg 0$, $\mathcal{G}_1 = \mathcal{G}'_1(l)$ and $\mathcal{F}_1 = \mathcal{G}_1 \oplus \mathcal{O}_{\mathbf{P}^N}^{n-3}$ are generated by global sections, their first four Chern classes are positive, while the fifth and following "odd" ones are negative.

References

[1] Fulton, W.: Intersection theory. Springer-Verlag, Berlin-Heidelberg (1984)

[2] Hartshorne, R.: Algebraic Geometry. GTM 52, Springer-Verlag, New York (1977)

[3] Hartshorne, R.: Stable reflexive sheaves. Mathematische Annalen 254, 121-176 (1980)

[4] Hartshorne, R.: Stable vector bundles of rank 2 on $\mathbf{P}^3$. Mathematische Annalen 238, 229-280 (1978)

[5] Huybrechts, D., Lehn, M.: The geometry of moduli spaces of sheaves. Aspects Math. E31, Bonn (1997).

[6] Kleiman, S. L.: The transversality of a general translate. Compositio Mathematica, 28 no. 3, p. 287-297, (1974).

[7] Martin-Deschamps, M., Perrin, D.: Sur la classification des courbes gauches, Astérisque 184-185, (1990).

[8] Matsumura H.: Commutative Algebra, Mathematics Lecture Notes Series, 56 (1980).

[9] Okonek, C.: Reflexive Garben auf $\mathbf{P}^4$. Mathematische Annalen 260, 211-237 (1982).